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ORIGINAL PAPER

The principal measure and distributional (p, q)-chaos of a coupled lattice system related with Belusov–Zhabotinskii reaction

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Abstract García Guirao and Lampart (J Math Chem 48:66–71, 2010; J Math Chem 2 48:159–164, 2010) said that for non-zero couplings constant, the lattice dynamical system is more complicated. Motivated by this, in this paper, we prove that this coupled lattice system is distributionally (p, q)-chaotic for any pair $0 \leq p \leq q \leq 1$ and its principal measure is not less than $\frac{2}{3} + \sum_{n=2}^{\infty} \frac{1}{n} \frac{2^{n-1}}{(2^n+1)(2^{n-1}+1)}$ for coupling constant $0 < \epsilon < 1$.

Keywords Coupled map lattice · Distributional (p, q)-chaos · Principal measure

1 Introduction

By a dynamical system, we mean a pair (X, f), where X is a compact metric space and $f : X \longrightarrow X$ is continuous. Since the introduction of the term of chaos in 1975 by Li and Yorke [13], known as Li–Yorke chaos today, dynamical properties were highly consider in the literature (see e.g., [2,6]) because are good examples of problems coming from the theory of topological dynamics and model many phenomena from biology, physics, chemistry, engineering and social sciences.

A very important generalization is that proposed by Schweizer and Smítal in [18], mainly because it is equivalent to positive topological entropy and some other concepts of chaos when restricted to the compact interval case [18] or hyperbolic symbolic spaces [15]. It is also remarkable that this equivalence does not transfer to higher

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dimensions, e.g. positive topological entropy does not imply distributional chaos in the case of triangular maps of the unit square [20] (the same happens when the dimension is zero [16]). Recently, we use an example to show that there exists a minimal system which exhibits distributional chaos in [24].

In many physical/ chemical engineering applications such as digital filtering, imaging and spatial dynamical system, dynamical systems have recently appeared as an important subject for investigation (see e.g., [5,12,17,21]). In [8,9], Guirao and Lampart studied a lattice dynamical system and proved it is chaotic in the sense of Li–Yorke, Devaney and positive entropy. Recently, we [23] proved that this system with non-zero coupling constant is chaotic in the sense of Li–Yorke and positive entropy. Motivated by these results, we shall further investigate into the dynamical properties of general lattice dynamical system.

2 Preliminaries

First, Let us recall some important notions of chaos such as Li–Yorke chaos, distributional chaos and positive topological entropy which is known to topological chaos. Throughout this paper, I denotes the unite closed interval [0, 1].

Definition 1 A pair of points $x, y \in X$ is called a Li–Yorke pair of system (X, f) if

(1) $\limsup_{n\to\infty} d(f^n(x), f^n(y)) > 0$

(2) $\liminf_{n\to\infty} d(f^n(x), f^n(y)) = 0.$

A subset $\mathcal{S} \subset X$ is called a scrambled set of f if $\sharp\mathcal{S} \geq 2$ and every pair of distinct points in $\mathcal{S}$ is a Li–Yorke pair. According to Li and Yorke [13], (X, f) is said to be chaotic in the sense of Li–Yorke if it has an uncountable scrambled set.

In 1986, Smítal [14] proved an important result for Li–Yorke chaos:

Proposition 1 *Dynamical system (I, f) is Li–Yorke chaotic if and only if it has a Li–Yorke pair.*

An attempt to measure the complexity of a dynamical system is based on a computation of how many points are necessary in order to approximate (in some sense) with their orbits all possible orbits of the system. A formalization of this intuition leads to the notion of topological entropy of the map f, which is due to Adler et al. [1]. We recall here the equivalent definition formulated by Bowen [7], and independently by Dinaburg [3]: the topological entropy of a map f is a number $h(f) \in [0, +\infty]$ defined by

$$h(f) = \lim_{\epsilon\to 0} \limsup_{n\to\infty} \sharp E(n, f, \epsilon)$$

where $E(n, f, \epsilon)$ is a (n, f, ϵ)-span with minimal possible number of points.

A map f is topologically chaotic if its topological entropy $h(f)$ is positive. Recently, the following result was proved by Blanchard et al. in [4].

Proposition 2 *If the topological entropy of a system (X, f) is positive then there exists a scrambled Cantor set, in particular f is chaotic in the sense of Li–Yorke.*

For any pair $(x, y) \in X \times X$ and for each $n \in \mathbb{N}$, the *distributional function* $F^n_{xy} : \mathbb{R}^+ \longrightarrow [0, 1]$ is defined by

$$F^n_{xy}(t, f) = \frac{1}{n}\sharp\left\{1 \le i \le n : \rho\left(f^i(x), f^i(y)\right) < t\right\}.$$

Define

$$F_{xy}(t, f) = \liminf_{n\to\infty} F^n_{xy}(t)$$
$$F^*_{xy}(t, f) = \limsup_{n\to\infty} F^n_{xy}(t)$$

Definition 2 Given $0 \le p \le q \le 1$, dynamical system (X, f) is distributionally (p, q)-chaotic if there exist an uncountable subset $\Gamma \subset X$ and $\epsilon > 0$ such that for any pair of distinct points $x, y \in \Gamma$, we have that $F_{xy}(t) = p$ and $F^*_{xy}(t) = q$ for all $0 < t < \epsilon$. Particularly, (X, f) is distributionally chaotic if it is distributionally $(0, 1)$-chaotic.

Recently, Yuan and Xiong [25] proved the following result:

Proposition 3 *The tent map* Λ *defined by* $\Lambda(x) = 1 - |1 - 2x|$, $x \in [0, 1]$ *is distributionally* (p, q)*-chaotic for any pair* $0 \le p \le q \le 1$.

In [19] the following measure for a dynamical (X, f) was introduced:

$$\mu_p(f) = \sup_{x,y\in X} \frac{1}{\mathrm{diam} X} \int_0^{+\infty} F^*_{xy}(t, f) - F_{xy}(t, f)\mathrm{d}t$$

They called it the principal measure of system (X, f). At the same time, they obtained that

$$\mu_p(\Lambda) = \frac{2}{3} + \sum_{n=2}^{\infty} \frac{1}{n} \frac{2^{n-1}}{(2^n + 1)(2^{n-1} + 1)}. \quad (1)$$

As is well known, it is a hard task to calculate this supremum; however, positiveness of $\mu_p(f)$ may be used as a preliminary test for chaos. This approach is more nice than that of topological entropy where (usually easier to calculate) upper bound does not help to answer whether the system is chaotic or not. Recently, we [22] proved that the principal measure of a quantum harmonic oscillator is 1.

3 Applications

The state space of LDS (Lattice Dynamical System) is the set

$$\mathcal{X} = \left\{x : x = \{x_i\}, x_i \in \mathbb{R}^d, i \in \mathbb{Z}^D, \|x_i\| < \infty\right\},$$

where $d \geq 1$ is the dimension of the range space of the map of state x_i, $D \geq 1$ is the dimension of the lattice and the l^2 norm $\|x\|_2 = \left(\sum_{i \in \mathbb{Z}^D} : |x_i|^2\right)^{1/2}$ is usually taken ($|x_i|$ is the length of the vector x_i).

Now we deal with a system coming from a lattice dynamical system stated by Kaneko in [10,11] which is related to the Belusov–Zhabotinskii reaction:

$$x_n^{m+1} = (1-\epsilon)f(x_n^m) + \epsilon/2\left[f(x_{n-1}^m) + f(x_{n+1}^m)\right] \tag{2}$$

where m is discrete time index, n is lattice side index with system size L (i.e., $n = 1, 2, \ldots, L$), ϵ is coupling constant and $f(x)$ is the unimodal map on the unite closed interval $I = [0, 1]$, i.e., $f(0) = f(1) = 0$ and f has unique critical point c with $0 < c < 1$.

In general, one of the following periodic boundary conditions of the system (2) is assumed:

(1) $x_n^m = x_{n+L}^m$,
(2) $x_n^m = x_n^{m+L}$,
(3) $x_n^m = x_{n+L}^{m+L}$,

standardly, the first case of the boundary conditions is used.

Let d be the product metric on the product space I^L, i.e.,

$$d((x_1, \ldots, x_L), (y_1, \ldots, y_L)) = \sqrt{\sum_{i=1}^{L} |x_i - y_i|^2}$$

for any $(x_1, \ldots, x_L), (y_1, \ldots, y_L) \in I^L$.

Define the map $F : (I^L, d) \longrightarrow (I^L, d)$ by $F(x_1, \ldots, x_L) = (y_1, \ldots, y_L)$ where $y_i = (1-\epsilon)f(x_i) + \epsilon/2\left[f(x_{i-1}) + f(x_{i+1}^m)\right]$. Clearly, system (2) is equivalent to system (I^L, F).

The Eq. (2) was studied by many authors, mostly experimentally or semi-analytically than analytically. In [8,9], the authors proved that if $f = \Lambda$, then coupled map lattice system is chaotic in the sense of Li–Yorke, Devaney and positive topological entropy for zero coupling constant. Now we shall further investigate into this system.

Theorem 1 *Given* $0 \leq p \leq q \leq 1$, *the system*

$$x_n^{m+1} = (1-\epsilon)f(x_n^m) + \epsilon/2\left[f(x_{n-1}^m) + f(x_{n+1}^m)\right] \tag{3}$$

is distributionally (p, q)*-chaotic for any* $0 < \epsilon < 1$ *when* $f = \Lambda$.

Proof By Proposition 3, it follows that Λ is distributionally (p, q)-chaotic, i.e., there exist an uncountable subset $\Gamma \subset I$ and $\epsilon > 0$ such that for any pair of distinct points $x, y \in \Gamma$ and any $0 < t < \epsilon$, we have

$$F_{xy}(t, \Lambda) = p \tag{4}$$

and

$$F^*_{xy}(t, \Lambda) = q. \tag{5}$$

Set $\mathscr{T} = \{(x_1, x_2, \ldots, x_L) \in I^L : x_1 = x_2 = \cdots x_L \in \Gamma\}$.

For any pair of distinct points $\overrightarrow{x} = (x, \ldots, x)$, $\overrightarrow{y} = (y, \ldots, y) \in \mathscr{T}$ and any positive integer n, it is easy to see that

$$F^n(\overrightarrow{x}) = (\Lambda^n(x), \ldots, \Lambda^n(x)) \tag{6}$$

and

$$F^n(\overrightarrow{y}) = (\Lambda^n(y), \ldots, \Lambda^n(y)) \tag{7}$$

Combining this with Eqs. (4) and (5), it follows that for any $0 < t < \sqrt{L}\epsilon$,

$$\begin{aligned} F_{\overrightarrow{x}\overrightarrow{y}}(t, F) &= \liminf_{n\to\infty} \frac{1}{n} \sharp \left\{1 \le i \le n : d(F^i(\overrightarrow{x}), F^i(\overrightarrow{y})) < t\right\} \\ &= \liminf_{n\to\infty} \frac{1}{n} \sharp \left\{1 \le i \le n : |\Lambda^i(x) - \Lambda^i(y)| < \frac{t}{\sqrt{L}}\right\} \\ &= F_{xy}\left(\frac{t}{\sqrt{L}}, \Lambda\right) = p, \end{aligned} \tag{8}$$

and

$$\begin{aligned} F^*_{\overrightarrow{x}\overrightarrow{y}}(t, F) &= \limsup_{n\to\infty} \frac{1}{n} \sharp \left\{1 \le i \le n : d(F^i(\overrightarrow{x}), F^i(\overrightarrow{y})) < t\right\} \\ &= \limsup_{n\to\infty} \frac{1}{n} \sharp \left\{1 \le i \le n : |\Lambda^i(x) - \Lambda^i(y)| < \frac{t}{\sqrt{L}}\right\} \\ &= F^*_{xy}\left(\frac{t}{\sqrt{L}}, \Lambda\right) = q. \end{aligned} \tag{9}$$

Hence, (I^L, F) [or, system (3)] is distributionally (p, q)-chaotic as $\mathscr{T}$ is uncountable.

Theorem 2 *For any* $0 < \epsilon < 1$, *the principal measure of the system*

$$x_n^{m+1} = (1 - \epsilon) f(x_n^m) + \epsilon/2 \left[f(x_{n-1}^m) + f(x_{n+1}^m)\right] \tag{10}$$

is not less than $\mu_p(f)$.

Proof For any pair of points $x, y \in I$, choose $\overrightarrow{x} = (x, \ldots, x)$ and $\overrightarrow{y} = (y, \ldots, y) \in I^L$. According to (8) and (9), it follows that for any $t > 0$,

$$F_{xy}(t, f) = F_{\overrightarrow{x}\overrightarrow{y}}(\sqrt{L}t, F) \tag{11}$$

and

$$F^*_{xy}(t, f) = F^*_{\overrightarrow{x}\overrightarrow{y}}(\sqrt{L}t, F). \tag{12}$$

So

$$\mu_p(F) \geq \sup_{x,y\in I} \frac{1}{\text{diam} I^L} \int_0^{+\infty} F^*_{\vec{x}\,\vec{y}}(t, F) - F_{\vec{x}\,\vec{y}}(t, F)\text{d}t$$

$$= \sup_{x,y\in I} \frac{1}{\sqrt{L}} \int_0^{+\infty} F^*_{xy}\left(\frac{t}{\sqrt{L}}, f\right) - F_{xy}\left(\frac{t}{\sqrt{L}}, f\right)\text{d}t$$

$$= \sup_{x,y\in I} \frac{1}{|I|} \int_0^{+\infty} F^*_{xy}(t, f) - F_{xy}(t, f)\text{d}t = \mu_p(t). \quad (13)$$

Remark (1) Applying Theorem 2 and Eq. (1), we have that the principal measure of system (2) is not less than $\frac{2}{3} + \sum_{n=2}^{\infty} \frac{1}{n} \frac{2^{n-1}}{(2^n+1)(2^{n-1}+1)}$ when $f = \Lambda$.

(2) According to [23, Theorem 1] and [18], similarly to the proof of Theorem 1, it follows that for each unimodal map f, system (2) is distributionally chaotic. This means that the principal measure of system (2) is positive.

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